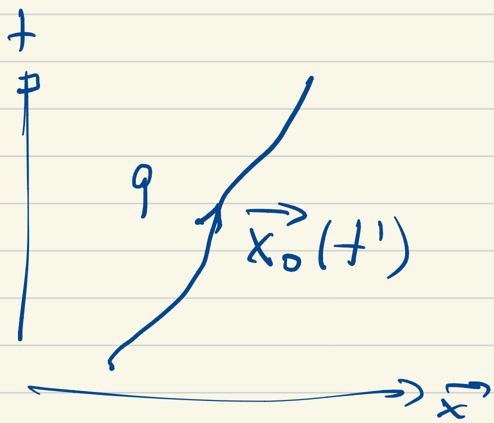
Lecture 6

- Radiation of a moving charge (cont.)
  - physical properties
- Multipole expansion
  - Electrostatics
  - Multipole moments
  - Magnetostatics

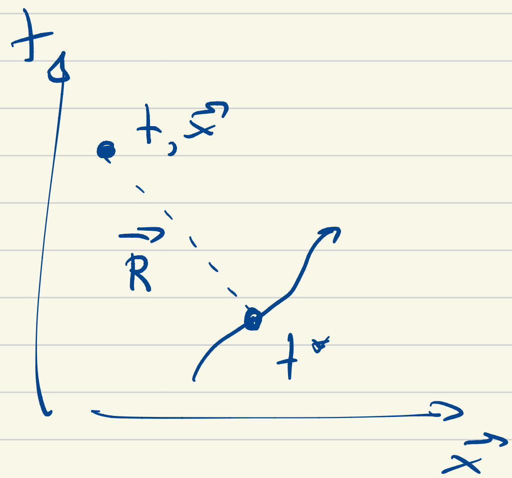
From Lecture 5:



$$\vec{E}(\vec{x}) =$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{(1-\vec{\beta}\cdot\vec{n})^3} \frac{1}{cR} \vec{n} \times \left([\vec{n} - \vec{\beta}] \times \dot{\vec{\beta}} \right) \Big|_{\text{all at } t_*} + O\left(\frac{1}{R^2}\right)$$

$$\vec{R} = \vec{x} - \vec{x}_0(t_*)$$



$t_*(t, \vec{x})$:

$$c(t - t_*) = R(t_*)$$

$$\vec{B} = \frac{1}{c}[\vec{n} \times \vec{R}]$$

$$\frac{d\mathcal{E}}{d\Omega\, dt} = \lim_{R\to\infty} R^2 \vec{S} \cdot \vec{n} \qquad \vec{S} = \frac{1}{\mu_0}[\vec{E} \times \vec{B}]$$

moving charge : $|R - \vec{\beta}\cdot\vec{R}|$ → is the distance to the charge at moment t (for constant velocity).

various β - dependent factors come from "changing the frame".

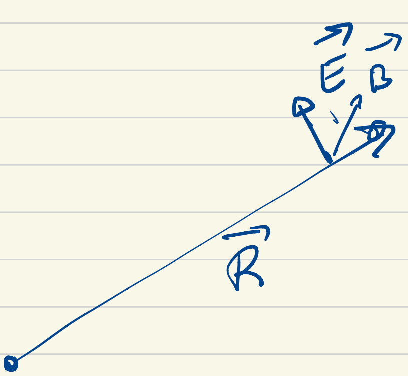
- The $\frac{1}{R}$ term is the radiation

$$\frac{q}{4\pi\varepsilon_0}\frac{1}{(1-\vec{\beta}\cdot\vec{n})^3}\frac{1}{cR}\,\vec{n}\times[\vec{n}-\vec{\beta}]\times\dot{\vec{\beta}}$$

the energy flux:

$$\frac{d\mathcal{E}}{d\Omega\, dt} = \lim_{R\to\infty} R^2\vec{S}\cdot\vec{n} \quad , \qquad \vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$\frac{1}{\mu_0}\left(\vec{E}\times\vec{n}\times\vec{E}\right)\cdot\vec{n} = \frac{E^2}{\mu_0}$$



$$\frac{1}{c\mu_0}\lim_{R\to\infty} R^2|\vec{E}|^2 =$$

$$= \frac{q^2}{16\pi^2 \underbrace{\varepsilon_0^2\mu_0 c^3}_{\varepsilon_0 c}} \frac{\left(\vec{n}\times(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)^2}{(1-\vec{n}\cdot\vec{\beta})^6}$$

• Non-relativistic limit:

$|\vec{\beta}| \ll 1 \qquad \left[\vec{\beta} = \frac{\vec{v}}{c} \right] \qquad \nearrow \dot{\vec{\beta}}$

$$\frac{d\mathcal{E}}{d\Omega\, dt} = \frac{q^2}{16\pi^2 \varepsilon_0 c} \left| \vec{n} \times \dot{\vec{\beta}} \right|^2$$

$$\frac{dP}{dt} = \int d\Omega \sin^2\theta \, \frac{q^2}{16\pi^2 \varepsilon_0 c} |\dot{\vec{\beta}}|^2 =$$

$$d\Omega = \int_0^{\pi} d\varphi \sin\theta \, d\theta$$

$$\left[\int_0^{\pi} \sin^3\theta = \frac{4}{3} \right] \qquad = \frac{q^2 |\dot{\vec{\beta}}|^2}{6\pi^2 \varepsilon_0 c}$$

Larmor Formula

In the relativistic case it is important to distinguish the radiation emitted in the frame of the particle:

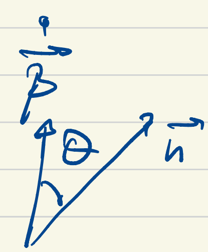
$$\frac{d\mathcal{E}}{d\Omega\, dt'} = \frac{d\mathcal{E}}{d\Omega\, dt} \cdot \frac{dt}{dt'} \qquad \frac{dt'}{dt} = \frac{1}{1-\vec{\beta}\cdot\vec{n}}$$

$$|\vec{\beta}| \simeq 1 \qquad 1-\vec{\beta}\cdot\vec{n} \longrightarrow 0$$

$$\vec{\beta} \parallel \vec{n}$$

• Rectilinear motion:

$$\beta \parallel \dot{\beta} \quad , \quad \frac{q^2}{16\pi^2 \varepsilon_0 c} \frac{\left(\vec{n}\times(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)^2}{(1-\vec{n}\cdot\vec{\beta})^6}$$



$$\frac{d\mathcal{E}}{d\Omega\, dt'} = \frac{q^2}{16\pi^2 \varepsilon_0 c} \frac{\dot{\vec{\beta}}^2 \sin^2\Theta}{\left(1-\beta\cos\Theta\right)^5}$$

total power: $$\frac{d\mathcal{E}}{dt'} = \frac{q^2 |\dot{\beta}|^2}{6\pi\varepsilon_0 c}\gamma^6 \quad , \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Derivation:

$$\int_0^{\pi} \frac{d\theta' \sin^3\theta}{(1-\beta\cos\theta)^5} = \int_{-1}^{1} \frac{dx\,(1-x^2)}{(1-\beta x)^5} = \frac{4}{3}\frac{1}{(1-\beta^2)^3}.$$

Consider

$\beta \to 1$. $\gamma \to \infty$, all radiation near $\theta \sim 0$

$$\gamma \sim \frac{1}{\sqrt{2(1-\beta)}} \;:\; 2\gamma^2 \sim \frac{1}{1-\beta}$$

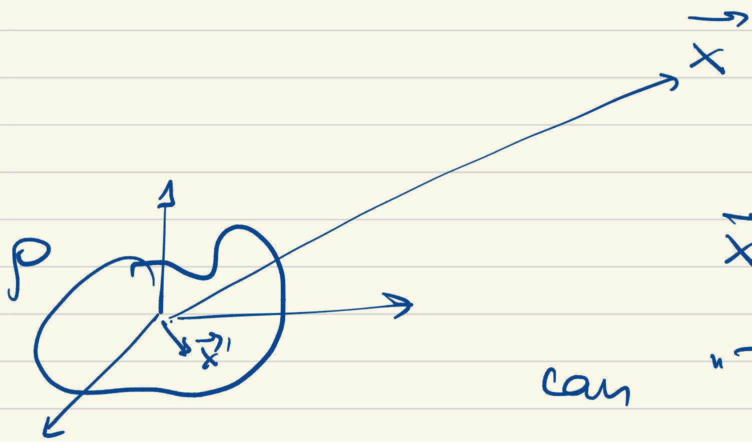
$$\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \to \frac{\theta^2}{\left(1-\beta\left(1-\frac{\theta^2}{2}\right)\right)} =$$

$$= \frac{\theta^2}{\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)^5} = \frac{2^5\gamma^{10}\theta^2}{(1+\gamma^2\theta^2)^5}$$

$$\frac{d\mathcal{E}}{d\Omega\, dt'} = \frac{2q^2}{\pi^2\varepsilon_0 c}|\dot{\vec{\beta}}|^2\gamma^8\frac{(\gamma\theta)^2}{(1+(\theta\gamma)^2)^5}$$

$(\theta\gamma)^2 = \frac{1}{4} \to$ max.

Multipole Expansion



$\vec{x} >> \vec{x}' \longrightarrow$ we can "Taylor expand" the internal structure.

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3\vec{x}' \, \frac{\rho(x')}{|\vec{x} - \vec{x}'|}$$

Let us start by Taylor expanding the Green's function:

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{n=0}^{\infty} \frac{1}{n!} x'_{i_1} \cdots x'_{i_n} \frac{\partial}{\partial x'_{i_1}} \cdots \frac{\partial}{\partial x'_{i_n}} \frac{1}{|\vec{x} - \vec{x}'|}\Bigg|_{\vec{x}'=0} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x'_{i_1} \cdots x'_{i_n}}{|\vec{x}|^{2n+1}} T_{i_1 \dots i_n}(\vec{x})$$

$$T_{i_1 \dots i_n}(\vec{x}) = |\vec{x}|^{2n+1} \frac{\partial}{\partial x'_{i_1}} \cdots \frac{\partial}{\partial x'_{i_n}} \frac{1}{|\vec{x} - \vec{x}'|} \Bigg|_{\vec{x}'=0}$$

T is symmetric and traceless:

$$T_{ii \dots i_n} \approx \Delta_{x'} \frac{1}{|\vec{x} - \vec{x}'|} \approx \delta(\vec{x} - \vec{x}') = 0$$

$$(\vec{x} >> \vec{x}')$$

$$T_{i_1 \dots i_n}(\vec{x}) = (2n-1)!! \, x_{i_1} x_{i_2} \cdots x_{i_n} - \\ - A_{i_1 \dots i_n},$$

where tensor A contains at least one kronecker δ_{ij}

$$T_i = x_i$$

$$T_{ij} = 3 x_i x_j - x^2 \delta_{ij}$$

We would like to expand the expression

$$\int d^3\vec{x}'\,\frac{\rho(x')}{|\vec{x}-\vec{x}'|} \qquad \text{at large } \frac{|\vec{x}|}{|\vec{x}'|}$$

but so far it is not clear which integrals of $\int \rho(\vec{x}')\, x'_{i_1} \dots x'_{i_n}$ will or will not contribute because of the structure of $T(\vec{x})$.

We then use the following trick:

$$x'_{i_1} x'_{i_2} \dots x'_{i_n}\, T_{i_1 \dots i_n}(\vec{x}) =$$

$A \sim \delta_{ij}$

$\dots \delta_{ij} \dots T_{\dots ij \dots} = 0$

$$= \left(x'_{i_1} x'_{i_2} \dots x'_{i_n} - \frac{A_{i_1 \dots i_n}(\vec{x}')}{(2n-1)!!} \right) T_{i_1 \dots i_n}(\vec{x}) =$$

$$= \frac{P_{i_1 \dots i_n}(\vec{x}')}{(2n-1)!!} \left(T_{i_1 \dots i_n}(\vec{x}) + A_{i_1 \dots i_n}(\vec{x}) \right)$$

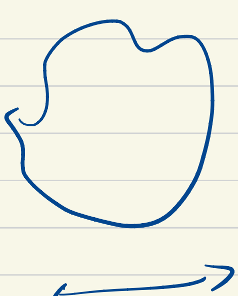
$$= P_{i_1 \ldots i_n}(\vec{x}')\, x_{i_1} \ldots x_{i_n}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3\vec{x}'\, \rho(x') \times$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x'_{i_1} \ldots x'_{i_n}}{|\vec{x}'|^{2n+1}} T_{i_1 \ldots i_n}(\vec{x}) =$$

$$= \frac{1}{4\pi\varepsilon_0} \int d^3\vec{x}' \sum_{n=0}^{\infty} \frac{1}{n!} \rho(x')\, T_{i_1 \ldots i_n}(\vec{x}')\, \frac{x_{i_1} \ldots x_{i_n}}{|\vec{x}|^{2n+1}} =$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{n!} Q_{i_1 \ldots i_n} \frac{x_{i_1} \ldots x_{i_n}}{|\vec{x}|^{2n+1}}$$



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$\Rightarrow Q_{i_1 \ldots i_n} \sim q\, a^n$ [since $\int_0^a dx'\, x'^n \sim a^n$]

n_{th} term goes as $\frac{q}{|\vec{x}|}\left(\frac{a}{|\vec{x}|}\right)^n$

Let us derive the corresponding electric fields:

$$\varphi^{(0)} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} \quad \Rightarrow \quad E^{(0)} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}$$

$$\varphi^{(1)} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{Q}^{(1)} \cdot \vec{x}}{|x|^3} \quad \Rightarrow$$

$$\vec{E} = -\vec{\nabla}\varphi$$

$$\vec{E}^{(1)} = \frac{1}{4\pi\varepsilon_0}\left(-\frac{\vec{Q}^{(1)}}{|x|^3} + \frac{3\vec{x}\,(\vec{Q}^{(1)} \cdot \vec{x})}{|x|^5} \right)$$

Magnetostatics

Since the Green's function for the vector potential is very similar we can use the similar techniques:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

We will not derive the all-orders expression, but just work at the leading orders:

$$A_i \approx \frac{\mu_0}{4\pi} \frac{1}{x} \int d^3x' \, J_i(x') +$$

$$+ \frac{x_j}{|\vec{x}|^3} \int d^3x' \, J_i(x') \, x'_j$$

Charge conservation: $\vec{\nabla} \cdot \vec{J} + \cancelto{0}{\dot{\rho}} = 0 \Rightarrow$

$$\vec{\nabla} \cdot \vec{J} = 0$$

$$\int_V x_i \partial_j J_j = 0 \Rightarrow \int_V \partial_j (x_i J_j) -$$

$$- \int_V J_i = 0 \Rightarrow \int_V J_i = 0$$

No magnetic monopoles! $\vec{\nabla} \cdot \vec{B} = 0$

We can also show that

$$\int_V d^3x \, (x_i J_j + x_j J_i) = 0$$

Dipole term:

$$x_j \int d^3x' \, J_i(x') \, x'_j =$$

$$= \frac{1}{2} x_j \int d^3x' \left(J_i(x') \, x'_j - J_j(x') \, x'_i \right) =$$

$$= \frac{1}{2} \int d^3x' \left(\vec{J} \, (\vec{x} \cdot \vec{x}') - \vec{x}' \left(\vec{J} \cdot \vec{x} \right) \right) =$$

$$\left[\vec{a} \times \vec{b} \times \vec{c} = \vec{b} \, (\vec{a} \cdot \vec{c}) - \vec{c} \, (\vec{a} \cdot \vec{b}) \right]$$

$$= \frac{1}{2}\int d^3x' \, \vec{x} \times \vec{J} \times \vec{x}' = \frac{1}{2}\underbrace{\Big(\int d^3x' \, \underbrace{\vec{x}' \times \vec{J}(x')}_{\vec{M}(x')}\Big)}_{\vec{m}} \times \vec{x}$$

$\vec{M}(x') \sim$ magnetisation

$\vec{m} \sim$ magnetic moment

$$\vec{A}^{(1)} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3} + \mathcal{O}(|x|^{-3})$$

$$\vec{B}^{(1)} = \vec{\nabla} \times \vec{A}^{(1)} = \dots = \frac{\mu_0}{4\pi} \cdot \frac{3\vec{n}(\vec{m}\cdot\vec{n}) - \vec{m}}{|\vec{x}|^3}$$

Note the similarity:

$$\vec{E}^{(1)} = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\vec{n}(\vec{Q}^{(1)}\cdot\vec{n}) - \vec{Q}^{(1)}}{|\vec{x}|^3}\right)$$